The Q-way of Doing Analysis

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Received: 23rd May 2005

The subject of q-calculus is a rich source of learning activities that address concept formation, exploration, variation, rich training and proof. This paper is a survey of the subject that shows how high school students using a computer algebra system can explore the field and prove some results.

1 INTRODUCTION

The impressive subject of calculus has a well known sister called *q*-calculus. It is best explored using a computer algebra system (CAS) to carry out the necessary calculations. In the course of learning this new subject one is automatically led to rethink the standard approach of calculus as well.

A mathematics teacher can achieve different goals using *q*-calculus. In the first part of the paper we show that the *q*-approach leads to an alternative definition of the derivative of a real valued function. This approach offers many problems that can be assigned to students to exercise limit based proof techniques.

In the second part we discard the limit process completely. The resulting expressions contain a free variable *q* that gives the subject its name.

The motivation to include this topic in a course on calculus comes not only from the subject itself but also from the educational processes it enables. The subject encourages exploration. As the theory runs parallel to standard calculus students can easily state conjectures and try to prove them. The new definitions are slight variations of the old definitions and further variations can be invented and their properties can be studied.

2 THE DERIVATIVE REVISITED

The standard approach of calculus leads to the definition of the derivative of a function as

$$f'(x) = \lim_{x_2 \to x} \frac{f(x_2) - f(x)}{x_2 - x}.$$

The fraction in this formula is interpreted as the slope of the secant and its limit defines the slope of the tangent. Usually, one introduces a variable h for the difference $h = x_2 - x$ to emphasise the quantity that tends to zero in the limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Of course, there is nothing wrong with this approach and the subsequent derivation of the various rules of calculus. From a teacher's perspective there is however the shortcoming that there is little opportunity for exercises on proving derivation rules. It seems hardly possible to invent any tractable assignments other than reproducing known proofs. Producing exercises of various degrees of difficulty requires variations of known situations, and precisely such a variation comes with the *q* approach.

It is not necessary to think of the difference between x_2 and x as an arithmetical difference. We may consider x^2 to be the *q*-fold of x, i.e. we may introduce a factor that relates these two variables: $x_2 = qx$. The limit in which the secant becomes the tangent is then given as q approaches one:

$$f'(x) = \lim_{q \to 1} \frac{f(qx) - f(x)}{qx - x}$$

As with the standard derivation students should take some time to experiment with numerical examples to gain a feeling for the new situation.

2.1 RULES OF DIFFERENTIATION

This alternative definition would not be of great interest if it wasn't possible to derive many rules of calculus. Let’s see how the use of a computer algebra system, such as MuPAD (www.mupad.com), can shed light on what is going on. Figure 1 shows some examples where this approach is used to obtain some derivatives.

```
qDIF:= (f,x) -> limit((subs(f,x=q*x)-f)/(q*x-x),q=1)
```

$$(f,x)\to \lim_{q\to 1}\frac{subs(f,x=q\cdot x)-f}{q\cdot x\ \ -x}$$

```
qDIF(x^2,x),qDIF(x^3,x),qDIF(x^4,x)
```

$$2\cdot x, 3\cdot x^2, 4\cdot x^3$$

```
qDIF(1/x,x)
```

$$-\frac{1}{x^2}$$

Figure 1 q-calculus applied to powers of x.

The new definition works as expected, but how can we gain more insight? We will remove the limit command and have a look at the quotients, see Figure 2.

```
Dq:= (f,x) -> (subs(f,x=q*x)-f)/(q*x-x)
```

$$(f,x)\to \frac{subs(f,x=q\cdot x)-f}{q\cdot x-x}$$

```
[i,factor(Dq(x^i,x))]$i=1..5
```

$$[1,1],[2,x\cdot(q+1)],[3,x^2\cdot(q+q^2+1)],[4,x^3\cdot(q+1)],[5,x^4\cdot(q+q^2+q^3+q^4+1)]$$

Figure 2 The quotients involved in the above q-calculus

This suggests that the denominators cancel out completely. This fact can be shown in general. Consider $f(x)=x^n$ with a natural number as exponent. Then

$$\begin{aligned} f'(x) &= \lim_{q\to 1}\frac{f(qx)-f(x)}{qx-x} \\ &= \lim_{q\to 1}\frac{q^n x^n - x^n}{x(q-1)} \\ &= x^{n-1}\lim_{q\to 1}\frac{q^n-1}{q-1} \end{aligned}$$

The last number can be reduced using the identity

$$q^n-1=(q-1)\cdot\left(q^{n-1}+q^{n-2}+....+q+1\right).$$

Hence the whole fraction reduces to

$$q^{n-1}+q^{n-2}+...+q+1$$

which clearly approaches n as $q\to 1$. For fractional exponents as in $f(x)=x^{1/n}$ almost the same reasoning applies:

$$\begin{aligned} f'(x) &= \lim_{q\to 1}\frac{q^{1/n}x^{1/n}-x^{1/n}}{x(q-1)} \\ &= x^{\frac{1}{n}-1}\lim_{t\to 1}\frac{t-1}{t^n-1} \end{aligned}$$

Here we applied the substitution $t=q^{1/n}$. Now, the same line of argument as before shows

$$f'(x)=x^{1/n-1}\lim_{t\to 1}\frac{t-1}{t^n-1}=\frac{x^{1/n-1}}{n}$$

The interplay of CAS exploration and rigorous proof can be repeated for several other types of functions. For logarithms a bit of work with MuPAD reveals a simple form of the quotient:

```
assume(q>0): factor(simplify(Dq(ln(x),x)))
```

In mathematical language:

$$\begin{aligned} \ln'(x) &= \lim_{q\to 1}\frac{\ln(qx)-\ln(x)}{qx-x} \\ &= \lim_{q\to 1}\frac{\ln(q)+\ln(x)-\ln(x)}{x(q-1)} \\ &= \frac{1}{x}\lim_{q\to 1}\frac{\ln(q)}{q-1}. \end{aligned}$$

This limit is easily determined using the CAS.

Trigonometric functions are also within the reach of MuPAD, as shown in Figure 3.

```
qDIF(sin(x),x)
```

$$\cos(x)$$

Figure 3 Differentiation of a trigonometric function

However, it is difficult to evaluate this limit using paper and pencil methods. Sum, product and quotient rules are derived in strictly analogous ways to the standard case. However, the somewhat different setup should make students think about every step rather than reproduce them from memory.

An interesting point is that the derivative of the absolute value function is rather straightforward in q-calculus as is shown below:

$$\frac{|qx|-|x|}{(q-1)x}=\frac{q|x|-|x|}{(q-1)x}=\frac{|x|}{x}$$

Taking a minute of thought reveals that at $x=0$ all of the above calculations have to be reconsidered. This shows the merits of the standard derivation very clearly.

3 FAREWELL TO THE LIMIT

During the above approach to the derivative we were led to define the analogue of the finite difference quotient, Dq in the MuPAD scripts. Now, we will take this quotient as our basic object of interest, i.e. we do not perform the limit. This is not too strange a thing to do if you recall that many results of calculus are applied at the end in numerical algorithms in which the derivative is again discretised. So, why not see how far one can get without performing limits at all?

In this spirit we define the q-differential of a function as

$$d_q f(x) := f(qx) - f(x),$$

which leads to

$$d_q x := qx - x = (q-1)x.$$

Combining, we arrive at the q-derivative we have met before:

$$D_q f(x) := \frac{d_q f(x)}{d_q x} := \frac{f(qx)-f(x)}{(q-1)x}.$$

The q-derivative of the power functions are

$$D_q x^n = \frac{q^n x^n - x^n}{x(q-1)} = x^{n-1}.\frac{q^n-1}{q-1}.$$

This motivates us to define the n-th q-number to be

$$[n]_q := \frac{q^n-1}{q-1}.$$

Then the powers q-differentiate almost as usual:

$$D_q x^n = [n]_q \cdot x^{n-1}.$$

4 *q*-NUMBERS

This definition calls for CAS explorations. The factorised forms of the first n q-numbers suggest that q-numbers have the same primes as standard numbers. Figure 4 shows the first 8 factorised q-numbers.

```
factor(qN(n))$n=1..8
```

$$1,\ q+1,\ q+q^2+1,\ (q+1)\cdot(q^2+1),\ q+q^2+q^3+q^4 1,$$
$$(q+1)\cdot(q+q^2+1)\cdot(-q+q^2+1),$$
$$(q+1)\cdot(q+q^2+1)\cdot(-q+q^2+1),$$
$$q+q^2+q^3+q^4+q^5+q^6+1,\ (q^2+1)\cdot(q^2+1)\cdot(q^4+1)$$

Figure 4 The first 8 factorised q-numbers

In certain formulas the version of the q-numbers used so far produces ugly results. Using a square root p of $q = p^2$ some results become more elegant when stated in terms of the "p-numbers":

$$(n)_p = \frac{p^n - p^{-n}}{p-p^{-1}} = \frac{p^{2n}-1}{p^2-1}\cdot\frac{1}{p^{n-1}} = [n]_q \cdot q^{\frac{1-n}{2}}$$

A warning note for the reader who consults the literature: this bracket notation is common in the field of quantum groups but bears the risk of confusion with the Pochhammer symbols.

For example, one can use the CAS to show that

$$(m+n)_p = p^n (m)_p + p^{-m}(n)_p$$
$$\frac{(n+1)_p + (n-1)_p}{(2)_p} = (n)_p$$

With these as examples the students can go on and discover and prove more identities using the CAS. Further common definitions involve the q-factorial $[n]! = [n]_p \ldots [1]_p$ and q-binomial coefficients built from these with the usual definition. They open up even more possibilities for CAS based discoveries.

However, the most important applications are in the field of noncommutative algebras. If two elements x, y of such an algebra satisfy the relation $y \cdot x = q \cdot x \cdot y$, one can apply the q-binomial theorem

$$(x+y)^n = \sum_{i=0}^{n} \frac{[n]!}{[i]![n-i]!} \cdot x^i \cdot y^{n-i} .$$

Although these noncommutative algebras are not difficult to handle they are far from typical of the subjects of the usual high school curriculum.

5 MORE ON Q-DIFFERENTIATION

The q-derivative obeys the usual rule for the sum of two functions:

$$D_q(f+g) = D_q(f) + D_q(g)$$

But what about the product rule? The calculation shown in Figure 5 proves the correct generalisation of the product rule:

```
A:=Dq(f(x)*g(x),x)
```

$$-\frac{f(q \cdot x) \cdot g(q \cdot x) - f(x) \cdot g(x)}{x - q \cdot x}$$

```
B:=Dq(f(x),x)*g(x)+f(q*x)*Dq(g(x),x)
```

$$\frac{f(q \cdot x) \cdot (g(x) - g(q \cdot x))}{x - q \cdot x} + \frac{g(x) \cdot (f(x) - f(q \cdot x))}{x - q \cdot x}$$

```
simplify(A-B)
```

$$0$$

Figure 5 Proving the product rule

Thus the product rule is

$$D_q(f(x) \cdot g(x)) = D_q(f(x)) \cdot g(x) + f(qx) \cdot D_q(g(x))$$

It seems that there is no easy generalisation of the chain rule. Of course, in q-mathematics there is not an exponential function but a q-exp function, with the following series definition:

$$\exp_q(x) := \sum_{n=0}^{\infty} \frac{1}{[n]!} \cdot x^n .$$

(This is a special case of what is called a q-hypergeometric series – a topic ideally suited for CAS, see Koepf (1998)).

It follows that $D_q(\exp_q(x)) = \exp_q(x)$, , which is far easier to prove by hand than with CAS and provides a good opportunity to reflect on the choice of the appropriate technique. Note however, that the result $\exp_q(x+y) = \exp_q(x) \cdot \exp_q(y)$ only holds if $y \cdot x = q \cdot x \cdot y$.

6 *q*-INTEGRATION

In the teaching of integration theory computer algebra offers many possibilities, e.g. Ben-Israel and Koepf (1994). (Here, it is one more example of concept formation and exploration.)

The inverse operation of q-differentiation shall be called q-integration (q-antiderivative would be more precise).

From

$$D_q(F(x)) = f(x)$$

it follows that

$$F(qx) - F(x) = (q-1) \cdot x \cdot f(x). .$$

Following Klimyk and Schmüdgen (1997), we substitute $q^i x$ for x and multiply by –1:

$$F(q^i \cdot x) - F(q^{i+1} \cdot x) = (1-q) \cdot q^i \cdot x \cdot f(q^i \cdot x)$$

Summing these equations for $i = 0,...,n-1$ gives:

$$F(x) - F(q^n \cdot x) = (1-q) \cdot x \cdot \sum_{i=0}^{n-1} q^i \cdot f(q^i \cdot x).$$

Klimyk and Schmüdgen impose the restriction that $0 < q < 1$ and assume F to be continuous at 0. Then the limit as $n \to \infty$ can be performed to yield an expression that can be turned into the definition of the q-Integral:

$$F(x) - F(0) = (1-q) \cdot x \cdot \sum_{i=0}^{\infty} q^i \cdot f(q^i \cdot x)$$

$$\int_0^c f(x) d_q x := c \cdot (1-q) \cdot \sum_{i=0}^{\infty} q^i \cdot f(q^i \cdot c).$$

This integral is in fact a Riemannian sum with the interval points $q^i.c$. We now define a MuPAD function for q-integration and check in a special case that the integral is in fact an antiderivative. This is illustrated in Figure 6.

```
assume(q<1): assume(q>0,_and):
qInt:= (f,x,c)->c*(1-q)*sum(q^i*subs(f,x=q^i*c),i=0..infinity);
```

$$(f, x, c) \to c \cdot (1-q) \cdot \left(\sum_{i=0}^{\infty} q^i \cdot subs(f, x = q^i \cdot c) \right)$$

```
simplify(qInt(x^2,x,a))
```

$$\frac{a^3}{q^2 + q + 1}$$

```
simplify(Dq(qInt(x^2,x,a),a))
```

$$a^2$$

Figure 6 An example of q-integration

Finally, we perform the limit $q \to 1$ and rediscover the classical integral, as shown in Figure 7.

```
f:=x: qI:=simplify(qInt(f,x,a)); limit(qI,q=1); int(f,x=0..a)
```

$$\frac{a^2}{q+1}$$

$$\frac{a^2}{2}$$

$$\frac{a^2}{2}$$

```
map([x,x^2,sqrt(x)], f→limit(simplify(qInt(f,x,a)),q=1))
```

$$\left[\frac{a^2}{2}, \frac{a^3}{3}, \frac{2\cdot a^{\frac{3}{2}}}{3}\right]$$

Figure 7 Classical integrals

With this we are led back to our starting point. Now we have a new limit approach to the classical antiderivative as well.

7 CONCLUSION

Computer algebra is well suited for supporting explorations and many (although not all) proofs in this area. While the subject is still evolving, in various branches there is already a rich body of knowledge as documented in the literature. We have concentrated on those parts that are within the reach of high school education. As the *q*-theory runs largely parallel to the standard theory students can carry over many ideas and create hypotheses and proofs. This style of working reflects very well the way mathematics is done at university. Many students think that mathematical research gives birth to complex new theories from scratch. However, in reality much more research work is conducted in a way that seeks to generalise and extend established theory. Hence, this subject lets students work almost like research mathematicians.

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BIOGRAPHICAL NOTES

Dr. Reinhard Oldenburg has studied mathematics, physics and computer science at Frankfurt and Göttingen where he wrote a PhD thesis on algebras connected to knot theory. Currently he is a teacher at the Felix-Klein-Gymnasium in Göttingen and is also engaged in teacher education at Göttingen University.